Quaternion Interpolation & The Arcball

COMP 4004
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(Re-)Evaluation

- A0 (10%) due Friday, Feb. 3, 4:00 pm
- A1 (15%) due Monday, Feb. 20, 4:00 pm
- A2 (15%) due Friday, Mar. 10, 4:00 pm
- A3 (20%) due Friday, Apr. 28, 4:00 pm
- Exam (40%)

Lemma 1

Assume $q = (w, x, y, z)$ is a unit quaternion.
Then there exists some angle θ and some unit vector $\vec{v}$ such that $q = (\cos\theta, \vec{v}\sin\theta)$.

Proof: Since $N(q) = w^2 + x^2 + y^2 + z^2 = 1$, we know that $-1 \le w \le 1$, so we let

$$\theta = \arccos(w)$$

$$\vec{v} = \left(\frac{x}{\sin\theta}, \frac{y}{\sin\theta}, \frac{z}{\sin\theta}\right)$$

Lemma 2

- For any *s≠0*, q and sq have the same action

$$\begin{aligned} p' &= (sq)p(sq)^{-1} \\ &= sqpq^{-1}s^{-1} \\ &= ss^{-1}qpq^{-1} \\ &= qpq^{-1} \end{aligned}$$

- I.e. *quaternions* are *homogeneous* in nature

Lemma 3

- Assume q is a unit quaternion. It's action on a scalar is:

$$\begin{aligned} qsq^{-1} &= sqq^{-1} \\ &= s \end{aligned}$$

Lemma 4

Assume q is a unit quaternion. It's action on a vector $\vec{u}$ is another vector $\vec{v}$, i.e. a quaternion $p = (0, \vec{v})$

Proof: Let $p = q\vec{u}q^{-1}$. What is it's scalar part p_w?

$$\begin{aligned} 2p_w &= p + p^* \\ &= q\vec{u}q^{-1} + (q\vec{u}q^{-1})^* \\ &= q\vec{u}q^* + (q\vec{u}q^*)^* \\ &= q\vec{u}q^* + q^{**}\vec{u}^*q^* \end{aligned}$$

$$\begin{aligned} 2p_w &= q\vec{u}q^* + q\vec{u}^*q^* \\ &= q(\vec{u} + \vec{u}^*)q^* \\ &= q0q^* \\ &= 0 \end{aligned}$$

Lemma 5

Let $p=(w,x,y,z)$ be a point in 3-D space in homogeneous coordinates, and let q be any unit quaternion. Then q's *action* on p, $p'=qpq^{-1}$, takes $p=(w,\vec{u})$ to $p'=(w,\vec{v})$, with $N(\vec{v})=N(\vec{u})$.

Proof: Applying Lemmas 3 & 4, we get:

$$\begin{aligned} p' &= qpq^{-1} \\ &= q(w+\vec{u})q^{-1} \\ &= qwq^{-1}+q\vec{u}q^{-1} \\ &= w+\vec{v} \end{aligned}$$

$$\begin{aligned} N(\vec{v}) &= N(p')-w^2 \\ &= N(q)N(p)N(q^{-1})-w^2 \\ &= N(p)-w^2 \\ &= N(\vec{u}) \end{aligned}$$

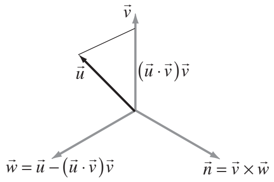
Theorem

Assume $q=(k\cos\theta, k\vec{v}\sin\theta)$ is any quaternion. Then the action of q on any homogeneous point $p=(w,\vec{u})$ rotates p around the axis $\vec{v}$ by 2θ.

Proof: By Lemma 1, assume that $\vec{v}$ is a unit vector. By Lemma 2, assume that $q=(\cos\theta,\vec{v}\sin\theta)$ is a unit quaternion. Because p is in homogeneous coordinates, assume $\vec{u}$ is a unit vector. Finally, by Lemma 5, we ignore w and consider $\vec{u}$.

Finding a Basis

Let $\vec{w}=\vec{u}-(\vec{u}\cdot\vec{v})\vec{v}$. This is perpendicular to $\vec{v}$ and coplanar with $\vec{u}$. For simplicity, assume that $\vec{w}$ is a unit vector.
Compute $\vec{n}=\vec{v}\times\vec{w}$ to get an ortho-normal basis $(\vec{v},\vec{w},\vec{n})$.
We will look at how q acts on $\vec{v}$ and $\vec{w}$ to prove our result.



Action on $\vec{v}$

$$\begin{aligned} q\vec{v}q^{-1} &= (\cos\theta,\sin\theta\vec{v})(0,\vec{v})(\cos\theta,-\sin\theta\vec{v}) \\ &= \left(0\cos\theta-(\sin\theta\vec{v})\cdot\vec{v},(\sin\theta\vec{v})\times\vec{v}+\cos\theta\vec{v}+0(\sin\theta\vec{v})\right)(\cos\theta,-\sin\theta\vec{v}) \\ &= \left(0-\sin\theta(\vec{v}\cdot\vec{v}),\sin\theta(\vec{v}\times\vec{v})+\cos\theta\vec{v}+\vec{0}\right)(\cos\theta,-\sin\theta\vec{v}) \\ &= \left(-\sin\theta(1),\sin\theta(\vec{0})+\cos\theta\vec{v}\right)(\cos\theta,-\sin\theta\vec{v}) \\ &= (-\sin\theta,\cos\theta\vec{v})(\cos\theta,-\sin\theta\vec{v}) \\ &= \left(-\sin\theta\cos\theta-(\cos\theta\vec{v})\cdot(-\sin\theta\vec{v}),(\cos\theta\vec{v})\times(-\sin\theta\vec{v})+(-\sin\theta)(-\sin\theta\vec{v})+\cos\theta(\cos\theta\vec{v})\right) \\ &= \left(-\sin\theta\cos\theta+\sin\theta\cos\theta(\vec{v}\cdot\vec{v}),\left(-\sin\theta\cos\theta(\vec{v}\times\vec{v})+\sin^2\theta\vec{v}+\cos^2\theta\vec{v}\right)\right) \\ &= \left(-\sin\theta\cos\theta+\sin\theta\cos\theta(1),\left(-\sin\theta\cos\theta(\vec{0})+\sin^2\theta\vec{v}+\cos^2\theta\vec{v}\right)\right) \\ &= \left(0,(\sin^2\theta+\cos^2\theta)\vec{v}\right) \\ &= (0,1\vec{v}) \\ &= (0,\vec{v}) \end{aligned}$$

Action on $\vec{w}$

$$\begin{aligned} q\vec{w}q^{-1} &= (\cos\theta,\sin\theta\vec{v})(0,\vec{w})(\cos\theta,-\sin\theta\vec{v}) \\ &= \left(0\cos\theta-(\sin\theta\vec{v})\cdot\vec{w},(\sin\theta\vec{v})\times\vec{w}+\cos\theta\vec{w}+0(\sin\theta\vec{v})\right)(\cos\theta,-\sin\theta\vec{v}) \\ &= \left(0-\sin\theta(\vec{v}\cdot\vec{w}),\sin\theta(\vec{v}\times\vec{w})+\cos\theta\vec{w}+\vec{0}\right)(\cos\theta,-\sin\theta\vec{v}) \\ &= \left(-\sin\theta(0),\sin\theta(\vec{n})+\cos\theta\vec{w}\right)(\cos\theta,-\sin\theta\vec{v}) \\ &= (0,\sin\theta\vec{n}+\cos\theta\vec{w})(\cos\theta,-\sin\theta\vec{v}) \\ &= \begin{pmatrix} 0\cos\theta-(\sin\theta\vec{n}+\cos\theta\vec{w})\cdot(-\sin\theta\vec{v}), \\ \quad (\sin\theta\vec{n}+\cos\theta\vec{w})\times(-\sin\theta\vec{v})+0(-\sin\theta\vec{v})+\cos\theta(\sin\theta\vec{n}+\cos\theta\vec{w}) \end{pmatrix} \\ &= \begin{pmatrix} -\sin^2\theta(\vec{n}\cdot\vec{v})-\sin\theta\cos\theta(\vec{w}\cdot\vec{v}), \\ \quad -\sin^2\theta(\vec{n}\times\vec{v})-\sin\theta\cos\theta(\vec{w}\times\vec{v})+0(-\sin\theta\vec{v})+\sin\theta\cos\theta\vec{n}+\cos^2\theta\vec{w} \end{pmatrix} \\ &= \left(-\sin^2\theta(0)-\sin\theta\cos\theta(0),-\sin^2\theta(\vec{w})-\sin\theta\cos\theta(-\vec{n})+\sin\theta\cos\theta\vec{n}+\cos^2\theta\vec{w}\right) \\ &= \left(0,(\cos^2\theta-\sin^2\theta)\vec{w}+(2\sin\theta\cos\theta)\vec{n}\right) \\ &= (0,\cos2\theta\vec{w}+\sin2\theta\vec{n}) \end{aligned}$$

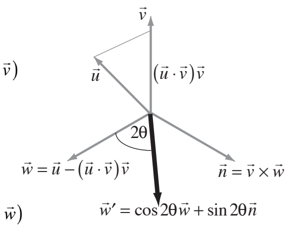
Interpretation

Note that $\vec{w}'$ is perpendicular to $\vec{v}$:

$$\begin{aligned} \vec{w}'\cdot\vec{v} &= (\cos2\theta\vec{w}+\sin2\theta\vec{n})\cdot\vec{v} \\ &= \cos2\theta(\vec{w}\cdot\vec{v})+\sin2\theta(\vec{n}\cdot\vec{v}) \\ &= \cos2\theta(0)+\sin2\theta(0) \\ &= 0 \end{aligned}$$

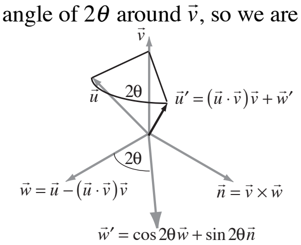
And the angle from $\vec{w}$ to $\vec{w}'$ is 2θ:

$$\begin{aligned} \vec{w}'\cdot\vec{w} &= (\cos2\theta\vec{w}+\sin2\theta\vec{n})\cdot\vec{w} \\ &= \cos2\theta(\vec{w}\cdot\vec{w})+\sin2\theta(\vec{n}\cdot\vec{w}) \\ &= \cos2\theta(1)+\sin2\theta(0) \\ &= \cos2\theta \end{aligned}$$



Action in General

We know that the $\vec{v}$ component of $\vec{u}$ is not changed, and that the $\vec{w}$ component is rotated by an angle of 2θ around $\vec{v}$, so we are done.



Uniqueness

A unit quaternion $q = (\cos\theta, \sin\theta\vec{v})$ is the *only* unit quaternion that gives the specified rotation. Moreover, the product of unit quaternions is another unit quaternion, so we can combine these rotations easily. Just like rotation matrices.

But quaternions can be interpolated more easily than matrices, and are more efficient and numerically stable.

Quaternion Est Demonstrandum

- So, quaternions rotate points
  - but they don't do translation
  - we still need matrices
  - and we need to convert to & fro
- But, first

Quaternion to Matrix

Let $q = (w, x, y, z)$ be a quaternion, and $p = (p_x, p_y, p_z, p_w)$ be a point in homogeneous coordinates. Note that with quaternions, we have been writing the w coordinate first, but in homogeneous coordinates, it comes last. This is why some authors put the w coordinate last in quaternions, but that leads to writing quaternions as $(\vec{v}, w)$ instead of $(w, \vec{v})$, which obscures the similarity to complex numbers.

Left-Multiplication

$$
\begin{aligned}
qp &= wp_w + wp_x i + wp_y j + wp_z k + xp_w i + xp_x i^2 + xp_y ij + xp_z ik \\
&\quad + yp_w j + yp_x ji + yp_y j^2 + yp_z jk + zp_w k + zp_x ki + zp_y kj + zp_z k^2 \\
&= (wp_x - zp_y + yp_z + xp_w) i \\
&\quad + (zp_x + wp_y - xp_z + yp_w) j \\
&\quad + (-yp_x + xp_y + wp_z + zp_w) k \\
&\quad + (-xp_x - yp_y + zp_z + wp_w) \\
&= \begin{bmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & z & w \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}
\end{aligned}
$$

We can write quaternion multiplication in matrix form: here's the left-multiplication, assuming a unit quaternion.

Left-Multiplication

$$
\begin{aligned}
pq^{-1} &= wp_w + wp_x i + wp_y j + wp_z k - xp_w i + xp_x i^2 + xp_y ij + xp_z ik \\
&\quad - yp_w j + yp_x ji + yp_y j^2 + yp_z jk - zp_w k + zp_x ki + zp_y kj + zp_z k^2 \\
&= (wp_x - zp_y + yp_z - xp_w) i \\
&\quad + (zp_x + wp_y - xp_z - yp_w) j \\
&\quad + (-yp_x + xp_y + wp_z - zp_w) k \\
&\quad + (xp_x + yp_y + zp_z + wp_w) \\
&= \begin{bmatrix} w & -z & y & -x \\ z & w & -x & -y \\ -y & x & w & -z \\ x & y & z & w \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}
\end{aligned}
$$

Here's the right-multiplication, assuming a unit quaternion.

Full Action

$$qpq^{-1} = \begin{bmatrix} w & -z & y & x \\ z & w & -x & y \\ -y & x & w & z \\ -x & -y & -z & w \end{bmatrix} \left(\begin{bmatrix} w & -z & y & -x \\ z & w & -x & -y \\ -y & x & w & -z \\ x & y & z & w \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix} \right)$$

$$= \begin{bmatrix} w^2+x^2-y^2-z^2 & 2(xy-wz) & 2(xz+wy) & 0 \\ 2(xy+wz) & w^2-x^2+y^2-z^2 & 2(yz-wx) & 0 \\ 2(xz-wy) & 2(yz+wx) & w^2-x^2-y^2+z^2 & 0 \\ 0 & 0 & 0 & w^2+x^2+y^2+z^2 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

$$= \begin{bmatrix} N(q)-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) & 0 \\ 2(xy+wz) & N(q)-2(x^2+z^2) & 2(yz-wx) & 0 \\ 2(xz-wy) & 2(yz+wx) & N(q)-2(x^2+y^2) & 0 \\ 0 & 0 & 0 & N(q) \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

$$= \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) & 0 \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) & 0 \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

Matrix to Quaternion

- Given a rotation matrix R, find:
  - the corresponding quaternion q, or:
  - the axis of rotation, v, and
  - the angle of rotation, θ

Solution I: Eigenvectors

A rotation matrix R has one eigenvalue of 1, whose eigenvector $\vec{v}$ is the axis of rotation. To find θ, let $\vec{u} = \vec{v} \times (0,0,1)$. If this is zero length, $\vec{v}$ is a multiple of $(0,0,1)$: use $(1,0,0)$ instead. Compute $\vec{u}' = R\vec{u}$, and take the dot product of $\vec{u}$ and $\vec{u}'$ to find $\cos\theta$

Solution II: the Matrix

Let $R = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) & 0 \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) & 0 \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

It therefore follows that:

$$\frac{3-a_{11}-a_{22}-a_{33}}{2} = x^2+y^2+z^2$$

$$\frac{a_{12}+a_{21}}{4} = xy$$

$$\frac{a_{13}+a_{31}}{4} = xz$$

$$\frac{a_{23}+a_{32}}{4} = yz$$

This is messy, but can be used to solve for $\vec{v} = (x,y,z)$. Of course, there is an easier way . . .

Solution III: the Hack

Choose any vector $\vec{u}$ (e.g. $\vec{u} = (0,0,1)$), and compute $\vec{u}' = R\vec{u}$. If $\vec{u}' = \pm\vec{u}$, then $\vec{u}$ is the axis of rotation. Otherwise, compute $\vec{v} = \vec{u} \times \vec{u}'$, which is perpendicular to both $\vec{u}$ and $\vec{u}'$, and must therefore be the axis of rotation. Now take the dot product of $\vec{u}$ and $\vec{u}'$ to find $\cos\theta$

Spherical Interpolation

- Quaternions rotate on *great circles*
- Assume that:
  - q defines the entire rotation
  - we want to interpolate in n steps

Exponential Solution

Let $q = (\cos\theta, \vec{v}\sin\theta)$ be the quaternion for the entire rotation.
We want to find a quaternion r such that applying it n times gives us a cumulative result of q:

$$\underbrace{rrrr\ldots rrr}_{n\text{ copies}}\, p\, \underbrace{r^{-1}r^{-1}r^{-1}r^{-1}\ldots r^{-1}r^{-1}r^{-1}}_{n\text{ copies}} = qpq^{-1}$$

$$r^n p (r^{-1})^n = qpq^{-1}$$

$$r = \sqrt[n]{q}$$

Because quaternions have multiplication and division, we can actually define and compute square roots, the exponential function and so on. However, it's messy, and nobody bothers.

Interpolation Hack

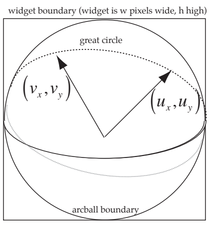
Let $q = (\cos\theta, \vec{v}\sin\theta)$ be the quaternion for the entire rotation.
We want to find a quaternion r such that applying it n times gives us a cumulative result of q:

$$\underbrace{rrrr\ldots rrr}_{n\text{ copies}}\, p\, \underbrace{r^{-1}r^{-1}r^{-1}r^{-1}\ldots r^{-1}r^{-1}r^{-1}}_{n\text{ copies}} = qpq^{-1}$$

Let $r = \left(\cos\frac{\theta}{n}, \vec{v}\sin\frac{\theta}{n}\right)$. Provided we have trig. functions or tables to compute θ, we're done!

Arcball Controller

- An arcball rotation consists of two mouse-clicks: *start* (*u*) and *end* (*v*)
- This gives a rotation along the great circle between *u* and *v*



2D to 3D

Both $\vec{u}$ and $\vec{v}$ are given in screen space, i.e. $\vec{u} = (u_x, u_y, 0)$, $\vec{v} = (v_x, v_y, 0)$. We scale and translate the coordinates into the range [-1..+1]:

$$u_x' = 2.0 - \frac{u_x}{0.5w},$$

$$u_y' = 2.0 - \frac{u_y}{0.5h}$$

and push any points outside the circle onto the boundary.
Now set $u_z = \sqrt{u_x^2 + u_z^2}$ to get a unit vector on the virtual trackball.
Do likewise with v_z.

Computing a Quaternion

Now set $u_z = \sqrt{u_x^2 + u_z^2}$ to get a unit vector on the virtual trackball.
Do likewise with v_z.
Let $q_u = (0, \vec{u})$ and $q_v = (0, \vec{v})$. Now define our rotation quaternion:

$$q = \sqrt{-q_v q_u} = \sqrt{-q_v}\sqrt{q_u}$$

Since $\vec{u}$ is a unit vector, $q_u^{-1} = q_u^* = (0, \vec{u})^* = (0, -\vec{u}) = -q_u$

$$\begin{aligned} qq_u q &= \left(\sqrt{-q_v q_u}\right) q_u \left(\sqrt{-q_v q_u}\right)^{-1} \\ &= \left(\sqrt{q_v}\sqrt{-q_u}\right) q_u \left(\sqrt{(-q_v q_u)^{-1}}\right) \\ &= \sqrt{q_v}\sqrt{q_u^{-1}}\, q_u \sqrt{q_u^{-1}}\sqrt{-q_v^{-1}} \\ &= \sqrt{q_v}(1)\sqrt{q_v} \\ &= q_v \end{aligned}$$

Arcball Version

You can save yourself some hassle here by using:

$$q = q_v q_u^{-1}$$

This will rotate the object by 2θ instead of θ, but is much easier to calculate. This is in fact what Ken Shoemake's arcball does for you, so your object spins twice as fast visually as the arcball. Why is this useful?

Keyframing

- Use the arcball to set joint rotations
- Set rotations at start and end frames
- Find the rotation quaternion
- Find the interpolation quaternion
- Compute the intermediate poses