Complex Numbers & Quaternions

COMP 4004
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Assignment Schedule

- A0 available Mon 16/01/06, due Mon 30/01/06, 4pm
- A1 available Mon 23/01/06, due Fri 10/02/06
- A2 available Mon 06/02/06, due Fri 24/02/06
- A3 available Mon 20/02/06, due Fri 10/03/06
- A4 available Mon 06/03/06, due Mon 10/04/06
- A5 available Wed 05/04/06, due Wed 19/04/06
- A6 available Wed 19/04/06, due Tues 02/05/06

Assignment Handin

- Hand in a tarball to my account:

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tar cvf A0_01234999.tar [files]
cp A0_01234999.tar ~hcarr/4004/A0
```

- Assignments due by 11:59 pm
- As always, include a readme.txt

Interpolating Angles

- Vectors / Matrices: can't interpolate
- Angle Interpolation: which angles?
  - Euler Angles: gimbal lock
  - Cardan Angles: angles not unique
  - Both: weird interpolations

Desiderata

- *Unique* representation of rotations
- Interpolates rotations cleanly
- Provides *great circle* arcs
- (Relatively) *simple* notation

Quaternions

- Homogeneous rotation coordinates
- Invented *before* scalars & vectors
  - vectors are a simplified version
- Based on complex numbers

Complex Numbers

- Numbers of the form:
  - *a*+*bi*
  - *a* is the *real* part
  - *b* is the *imaginary* part
  - $i = \sqrt{-1}$

Operations

Addition:

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i$$

Multiplication:

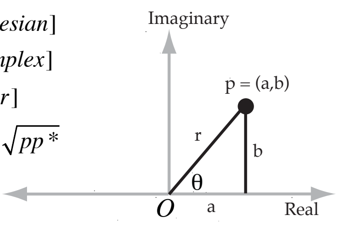
$$\begin{aligned}(a_1 + b_1 i)(a_2 + b_2 i) &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 (-1) \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i\end{aligned}$$

Conjugation

$$\begin{aligned}(a+bi)^* &= a - bi \\ (a+bi) + (a+bi)^* &= (a+bi) + (a-bi) \\ &= 2a \\ (a+bi)(a+bi)^* &= (a+bi)(a-bi) \\ &= (aa - b(-b)) + (ab + (-b)a) \\ &= a^2 + b^2\end{aligned}$$

Spatial Interpretation

$$\begin{aligned}p &= (a,b)\ [Cartesian] \\ &= a + bi\ [Complex] \\ &= (r,\theta)\ [Polar] \\ r &= \sqrt{a^2 + b^2} = \sqrt{pp^*} \\ \theta &= \arctan\left(\frac{b}{a}\right)\end{aligned}$$



Polar Multiplication

Let $a_3 + b_3 i = (a_1 + b_1 i)(a_2 + b_2 i)$

$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i$$

What is $(r_3, \theta_3) = ?$

$$\begin{aligned}r_3 &= \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + b_1 a_2)^2} \\ &= \sqrt{a_1^2 a_2^2 - 2 a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + a_1^2 b_2^2 + 2 a_1 a_2 b_1 b_2 + a_2^2 b_1^2} \\ &= \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \\ &= \sqrt{(a_1^2 + b_1^2)}\sqrt{(a_2^2 + b_2^2)} \\ &= r_1 r_2\end{aligned}$$

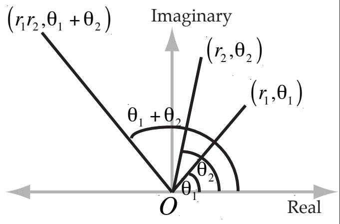
What about Theta?

$$\begin{aligned}\theta_3 &= \arctan\left(\frac{b_3}{a_3}\right) \\ &= \arctan\left(\frac{a_1 b_2 + a_2 b_1}{a_1 a_2 - b_1 b_2}\right) \\ &= \arctan\left(\frac{\frac{a_1 b_2 + a_2 b_1}{a_1 a_2}}{\frac{a_1 a_2 - b_1 b_2}{a_1 a_2}}\right)\end{aligned}$$

$$\begin{aligned}\theta_3 &= \arctan\left(\frac{\frac{b_2}{a_2} + \frac{b_1}{a_1}}{1 - \left(\frac{b_1}{a_1}\right)\left(\frac{b_2}{a_2}\right)}\right) \\ &= \arctan\left(\frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}\right) \\ &= \arctan(\tan(\theta_1 + \theta_2)) \\ &= \theta_1 + \theta_2\end{aligned}$$

Geometric Interpretation

- This gives us:
  - scaling (r)
  - rotation (θ)
  - all within complex multiplication



Extending to 3-D

- We'll look at complex conjugates:

$$(a+bi)(a+bi)^* = (a^2+b^2) = r^2$$

- Does this work for 3 coordinates?

$$\begin{aligned}(a+bi+cj)(a+bi+cj)^* &= (a+bi+cj)(a-bi-cj) \\ &= a^2 - abi - acj + abi - b^2i^2 - bcij + acj - bcji - c^2j^2 \\ &= a^2 - b^2i^2 - c^2j^2 - bcij - bcji \\ &= a^2 + b^2 + c^2 - bcij - bcji\end{aligned}$$

The Fourth Coordinate

- We need *bcij* and *bcji* to cancel out
- So we add a fourth coordinate:

$$k = ij = -ji$$
$$jk = j(-ji) = -j^2 i = -(-1)i = i = -kj$$
$$ki = (-ji)i = -ji^2 = -j(-1) = j = -ik$$

With Quaternions

$$\begin{aligned}(a+bi+cj+dk)(a+bi+cj+dk)^* &= (a+bi+cj+dk)(a-bi-cj-dk) \\ &= a^2 - abi - acj - adk + abi - b^2i^2 - bcij - bdik \\ &\quad +acj - bcji - c^2j^2 - cdjk + adk - bdki - cdkj - d^2k^2 \\ &= a^2 + b^2 + c^2 + d^2 \\ &\quad -abi + abi - cdjk - cdkj \\ &\quad -acj - bdik + acj - bdki \\ &\quad -adk - bcij - bcji + adk \\ &= a^2 + b^2 + c^2 + d^2\end{aligned}$$

- Now it all works!

Operations

Addition:

$$\begin{aligned}q_1 + q_2 &= (a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k) \\ &= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k\end{aligned}$$

Multiplication:

$$\begin{aligned}q_1 q_2 &= (a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) \\ &= a_1a_2 + a_1b_2 i + a_1c_2 j + a_1d_2 k + b_1a_2 i + b_1b_2 i^2 + b_1c_2 ij + b_1d_2 ik \\ &\quad +c_1a_2 j + c_1b_2 ji + c_1c_2 j^2 + c_1d_2 jk + d_1a_2 k + d_1b_2 ki + d_1c_2 kj + d_1d_2 k^2 \\ &= a_1a_2 + a_1b_2 i + a_1c_2 j + a_1d_2 k + b_1a_2 i + b_1b_2(-1) + b_1c_2(k) + b_1d_2(-j) \\ &\quad +c_1a_2 j + c_1b_2(-k) + c_1c_2(-1) + c_1d_2(i) + d_1a_2 k + d_1b_2(-j) + d_1c_2(-i) + d_1d_2(-1) \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)1 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\ &\quad +(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j + (a_1d_2 + b_1c_2 - c_1b_2 + ad)k\end{aligned}$$

Geometric Interpretation

- *i,j,k* are different from *1*:
  - *i* becomes *j* becomes *k* becomes *i*
  - looks like rotating between *x,y,z*
- In a quaternion *q=(w,x,y,z)*
  - *w* is the *scalar* part
  - *(x,y,z)* is the *vector* part

Notation

- Scalar s: $s = (s, \vec{0})$ $= (s, 0, 0, 0)$
- Vector v: $\vec{v} = (0, \vec{v})$ $= (0, x, y, z)$
- Quaternion q: $q = (w, \vec{v})$ $= (w, x, y, z)$

Properties

- Scalar multiplication: $sq = qs$ $= (s, \vec{0})(w, \vec{v}$ $= (sw, s\vec{v})$
- Associativity & Distributivity: $(pq)r = p(qr)$ $p(q + r) = pq + pr$
- Anti-commutativity: $(pq)^* = (q^* p^*)$

Conjugation

$$
\begin{aligned}
(w, x, y, z)^* &= (w, -x, -y, -z) \\
(w, \vec{v})^* &= (w, -\vec{v}) \\
(pq)^* &= q^* p^* \\
(w, \vec{v}) + (w, \vec{v})^* &= (w + w, \vec{v} + (-\vec{v})) \\
&= (2w, \vec{0})
\end{aligned}
$$

Vector Multiplication

$$
\begin{aligned}
\vec{v}_1 \vec{v}_2 &= (0 + x_1 i + y_1 j + z_1 k)(0 + x_2 i + y_2 j + z_2 k) \\
&= (0 - x_1 x_2 - y_1 y_2 - z_1 z_2) + (0 + 0 - y_1 z_2 + z_1 y_2) i \\
&\quad + (0 + x_1 z_2 + 0 - z_1 x_2) j + (0 - x_1 y_2 + y_1 z_2 + 0) k \\
&= (-\vec{v}_1 \cdot \vec{v}_2, \vec{v}_1 \times \vec{v}_2)
\end{aligned}
$$

- Dot product *and* cross product!

More Multiplication

$$
\begin{aligned}
q_1 q_2 &= (w_1, \vec{v}_1)(w_2, \vec{v}_2) \\
&= \left((w_1, \vec{0}) + (0, \vec{v}_1)\right)\left((w_2, \vec{0}) + (0, \vec{v}_2)\right) \\
&= (w_1, \vec{0})(w_2, \vec{0}) + (w_1, \vec{0})(0, \vec{v}_2) \\
&\quad + (0, \vec{v}_1)(w_2, \vec{0}) + (0, \vec{v}_1)(0, \vec{v}_2) \\
&= (w_1 w_2, \vec{0}) + (0, w_1 \vec{v}_2) + (0, w_2 \vec{v}_1) + (-\vec{v}_1 \cdot \vec{v}_2, \vec{v}_1 \times \vec{v}_2) \\
&= (w_1 w_2 - \vec{v}_1 \cdot \vec{v}_2, \vec{v}_1 \times \vec{v}_2 + w_1 \vec{v}_2 + w_2 \vec{v}_1)
\end{aligned}
$$

Norm & Inverse

$$
\begin{aligned}
N(q) &= qq^* = q^* q \\
&= w^2 + x^2 + y^2 + z^2 \\
&= w^2 + v \cdot v \\
N(pq) &= N(p) N(q) \\
N(q^*) &= N(q) \\
q^{-1} &= q^* / N(q)
\end{aligned}
$$

Lemma

Assume $q = (w, x, y, z)$ is a unit quaternion.

Then there exists some angle θ and some unit vector $\vec{v}$ such that $q = (\cos\theta, \vec{v}\sin\theta)$.

Proof: Since $N(q) = w^2 + x^2 + y^2 + z^2 = 1$, we know that $-1 \le w \le 1$, so we let

$\theta = \arccos(w)$

$$\vec{v} = \left(\frac{x}{\sin\theta}, \frac{y}{\sin\theta}, \frac{z}{\sin\theta}\right)$$

Quaternion Action

Let $p = (w, x, y, z)$ be a point in 3-D space in homogeneous coordinates, and let q be any non-zero quaternion.

Define q's *action* on p to be given by:

$p' = qpq^{-1}$

Lemma 2

- For any $s \neq 0$, q and sq have the same action

$$\begin{aligned} p' &= (sq)\, p\, (sq)^{-1} \\ &= sqpq^{-1}s^{-1} \\ &= ss^{-1}qpq^{-1} \\ &= qpq^{-1} \end{aligned}$$

- I.e. *quaternions* are *homogeneous* in nature

Lemma 3

- Assume q is a unit quaternion. It's action on a scalar is:

$$\begin{aligned} qsq^{-1} &= sqq^{-1} \\ &= s \end{aligned}$$

Lemma 4

Assume q is a unit quaternion. It's action on a vector $\vec{u}$ is another vector $\vec{v}$, i.e. a quaternion $p = (0, \vec{v})$

Proof: Let $p = q\vec{u}q^{-1}$. What is it's scalar part p_w?

$$\begin{aligned} 2p_w &= p + p^* \\ &= q\vec{u}q^{-1} + (q\vec{u}q^{-1})^* \\ &= q\vec{u}q^* + (q\vec{u}q^*)^* \\ &= q\vec{u}q^* + q^{**}\vec{u}^* q^* \end{aligned}$$

$$\begin{aligned} 2p_w &= q\vec{u}q^* + q\vec{u}^* q^* \\ &= q(\vec{u} + \vec{u}^*)q^* \\ &= q0q^* \\ &= 0 \end{aligned}$$

Lemma 5

Let $p = (w, x, y, z)$ be a point in 3-D space in homogeneous coordinates, and let q be any unit quaternion. Then q's *action* on p, $p' = qpq^{-1}$, takes $p = (w, \vec{u})$ to $p' = (w, \vec{v})$, with $N(\vec{v}) = N(\vec{u})$.

Proof: Applying Lemmas 3 & 4, we get:

$$\begin{aligned} p' &= qpq^{-1} \\ &= q(w + \vec{u})q^{-1} \\ &= qwq^{-1} + q\vec{u}q^{-1} \\ &= w + \vec{v} \end{aligned}$$

$$\begin{aligned} N(\vec{v}) &= N(p') - w^2 \\ &= N(q)N(p)N(q^{-1}) - w^2 \\ &= N(p) - w^2 \\ &= N(\vec{u}) \end{aligned}$$

Theorem

Assume $q = (k\cos\theta, k\vec{v}\sin\theta)$ is any quaternion.
Then the action of q on any homogeneous point $p = (w, x, y, z)$ rotates p around the axis $\vec{v}$ by 2θ.
Proof: By Lemma 2, we can ignore the constant k, so we assume that $q = (\cos\theta, \vec{v}\sin\theta)$ is a unit quaternion.